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Solutions of a discretized Toda field equation for D_r from analytic Bethe ansatz

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Abstract. Commuting transfer matrices of $U_q(X_r^{(1)})$ vertex models obey the functional relations which can be viewed as an X_r type Toda field equation on discrete spacetime. Based on analytic Bethe ansatz we present, for $X_r = D_r$, a new expression of its solution in terms of determinants and Pfaffians.

1. Introduction

In Kuniba *et al* (1994), a family of functional relations, a T-system, was proposed for commuting transfer matrices of solvable lattice models associated to any quantum affine algebras $U_q(X_r^{(1)})$. For $X_r = D_r$ it reads as follows:

$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(a-1)}(u)T_m^{(a+1)}(u) \qquad 1 \leqslant a \leqslant r-3 \tag{1.1a}$$

$$T_m^{(r-2)}(u-1)T_m^{(r-2)}(u+1) = T_{m+1}^{(r-2)}(u)T_{m-1}^{(r-2)}(u) + T_m^{(r-3)}(u)T_m^{(r-1)}(u)T_m^{(r)}(u) \tag{1.1b}$$

$$T_m^{(a)}(u-1)T_m^{(a)}(u+1) = T_{m+1}^{(a)}(u)T_{m-1}^{(a)}(u) + T_m^{(r-2)}(u) \qquad a = r-1, r \tag{1.1c}$$

where $T_m^{(a)}(u)$ ($m \in \mathbb{Z}, u \in \mathbb{C}$: spectral parameters) denote the transfer matrices with the auxiliary space labelled by a and m. We shall employ the boundary condition $T_{-1}^{(a)}(u) = 0, T_0^{(a)}(u) = 1$, which is natural for the transfer matrices. Then, solving (1.1) successively, one can express $T_m^{(a)}(u)$ uniquely as a polynomial of the fundamental polynomials $T_1^{(1)}, \ldots, T_1^{(r)}$. The aim of this paper is to give a new expression to the solution of (1.1) motivated by the analytic Bethe ansatz (Reshetikhin 1983). There is an earlier solution in Kuniba *et al* (1996), which is expressed only by the fundamental polynomials $T_1^{(1)}, \ldots, T_1^{(r)}$. However, in this paper we begin by introducing the auxiliary transfer matrix (or 'dress function' in the analytic Bethe ansatz) $\mathcal{T}^a(u)$ (2.10) for any $a \in \mathbb{Z}$ and establish a new functional relation as in proposition 2.3 (see later). For $1 \leqslant a \leqslant r-2$, $\mathcal{T}^a(u)$ is just $T_1^{(a)}(u)$ while for $a \geqslant r-1$ it is quadratic in $T_1^{(r)}$ and $T_1^{(r-1)}$. We then express the solution as the determinants and Pfaffians with matrix elements $0, \pm\mathcal{T}^a, \pm T_1^{(r-1)}$ or $\pm T_1^{(r)}$. Moreover those determinants and Pfaffians are taken over the matrices with dense distributions of non-zero elements as opposed to the sparse ones in Kuniba *et al* (1996).

The two types of representation of the solutions obtained here and in Kuniba *et al* (1996) are significant in their own right. The sparse type (Kuniba *et al* 1996), arises straightforwardly from a manipulation of the T-system only. On the other hand, the

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dense type is more connected with the analytic Bethe ansatz idea (Kuniba and Suzuki 1995), in view of which it is most natural to introduce the $\mathcal{T}^a$ as well as the Q-functions $Q_1(u), \ldots, Q_r(u)$. It should be noted that $T_m^{(a)}(u)$ in this paper is a solution of (1.1) for arbitrary Q-functions. The definition through the Bethe equations as in (2.1) to (2.2) is needed only when one requires $T_m^{(a)}(u)$ to yield the actual transfer matrix spectra. We note that two similar such representations are also available for the solution of the B_r T-system in Kuniba *et al* (1995) and Kuniba *et al* (1996).

As the previous cases (Kuniba *et al* 1995, Kuniba *et al* 1996), all the proofs of the determinant and Pfaffian formulae reduce essentially to the Jacobi identity (4.1) (see later), a well known machinery in soliton theories. In fact, it was first pointed out in (Kuniba *et al* 1995) that the T-system for $U_q(X_r^{(1)})$ may be viewed as a Toda field equation (Leznov and Saveliev 1979, Mikhailov *et al* 1981) with discrete spacetime variables u and m. Mathematically, it implies a common structure between discretized soliton equations (Ablowitz and Ladik 1976, Date *et al* 1982, Hirota 1977) and representation rings of finite-dimensional modules over Yangians or quantum affine algebras. Our new solution here exemplifies such an interplay further. See Krichever *et al* (1996) for a similar perspective.

The outline of the paper is as follows. Section 2 is a brief review of the D_r case of the analytic Bethe ansatz results (Kuniba and Suzuki 1995). For $T_1^{(1)}(u)$, $T_1^{(r)}(u)$ and $T_1^{(r-1)}(u)$ it partially overlaps the earlier result in Reshetikhin (1983). Proposition 2.3 is new and plays a key role in the subsequent arguments. In section 3 we present the solution, which is proved in section 4. Section 5 is devoted to a discussion. The appendix provides a number of formulae similar to those used in section 4.

2. Review of the results for fundamental representations

In this section we basically follow (Kuniba and Suzuki 1995). Let $\{\alpha_1, \ldots, \alpha_r\}$ be the simple roots normalized so that $(\alpha_a|\alpha_b) =$ Cartan matrix. The Bethe ansatz equation reads

$$-1 = \prod_{b=1}^{r} \frac{Q_b(u_k^{(a)} + (\alpha_a|\alpha_b))}{Q_b(u_k^{(a)} - (\alpha_a|\alpha_b))} \qquad 1 \leqslant a \leqslant r,\ 1 \leqslant k \leqslant N_a \tag{2.1}$$

$$Q_a(u) = \prod_{j=1}^{N_a} [u - u_j^{(a)}] \tag{2.2}$$

where $[u] = (q^u - q^{-u})/(q - q^{-1})$ and $N_a \in \mathbb{Z}_{\geqslant 0}$. In this paper, we suppose that q is generic. We define a set

$$J = \{1, 2, \ldots, r, \bar{r}, \ldots, \bar{1}\} \tag{2.3}$$

with the partial order

$$1 \prec 2 \prec \cdots \prec r-1 \prec \begin{matrix} r \\ \bar{r} \end{matrix} \prec \overline{r-1} \prec \cdots \prec \bar{2} \prec \bar{1}. \tag{2.4}$$

Note that there is no order between r and $\bar{r}$. For $a \in J$, set

$$z(a;u) = \frac{Q_{a-1}(u+a+1)Q_a(u+a-2)}{Q_{a-1}(u+a-1)Q_a(u+a)} \qquad \text{for } 1 \leqslant a \leqslant r-2$$

$$z(r-1;u) = \frac{Q_{r-2}(u+r)Q_{r-1}(u+r-3)Q_r(u+r-3)}{Q_{r-2}(u+r-2)Q_{r-1}(u+r-1)Q_r(u+r-1)}$$

$$z(r;u) = \frac{Q_{r-1}(u+r+1)Q_r(u+r-3)}{Q_{r-1}(u+r-1)Q_r(u+r-1)}$$

$$z(\bar{r}; u) = \frac{Q_{r-1}(u+r-3)Q_r(u+r+1)}{Q_{r-1}(u+r-1)Q_r(u+r-1)} \tag{2.5}$$
$$z(\overline{r-1}; u) = \frac{Q_{r-2}(u+r-2)Q_{r-1}(u+r+1)Q_r(u+r+1)}{Q_{r-2}(u+r)Q_{r-1}(u+r-1)Q_r(u+r-1)}$$
$$z(\bar{a}; u) = \frac{Q_{a-1}(u+2r-a-3)Q_a(u+2r-a)}{Q_{a-1}(u+2r-a-1)Q_a(u+2r-a-2)} \qquad \text{for } 1 \leqslant a \leqslant r-2$$

where $Q_0(u) = 1$. For $(\xi_1, \ldots, \xi_r) \in \{\pm\}^r$, we define the function $sp(\xi_1, \ldots, \xi_r; u)$ recursively by

$$sp(+, +, \xi_3, \ldots, \xi_r; u) = \tau^Q sp(+, \xi_3, \ldots, \xi_r; u)$$
$$sp(+, -, \xi_3, \ldots, \xi_r; u) = \frac{Q_1(u+r-3)}{Q_1(u+r-1)}\tau^Q sp(-, \xi_3, \ldots, \xi_r; u)$$
$$sp(-, +, \xi_3, \ldots, \xi_r; u) = \frac{Q_1(u+r+1)}{Q_1(u+r-1)}\tau^Q sp(+, \xi_3, \ldots, \xi_r; u+2)$$
$$sp(-, -, \xi_3, \ldots, \xi_r; u) = \tau^Q sp(-, \xi_3, \ldots, \xi_r; u+2) \tag{2.6}$$

with the following initial conditions:

$$sp(+,+,+,+;u) = \frac{Q_4(u-1)}{Q_4(u+1)}$$
$$sp(+,+,-,-;u) = \frac{Q_2(u)Q_4(u+3)}{Q_2(u+2)Q_4(u+1)}$$
$$sp(+,-,+,-;u) = \frac{Q_1(u+1)Q_2(u+4)Q_3(u+1)}{Q_1(u+3)Q_2(u+2)Q_3(u+3)}$$
$$sp(+,-,-,+;u) = \frac{Q_1(u+1)Q_3(u+5)}{Q_1(u+3)Q_3(u+3)}$$
$$sp(-,+,+,-;u) = \frac{Q_1(u+5)Q_3(u+1)}{Q_1(u+3)Q_3(u+3)} \tag{2.7}$$
$$sp(-,+,-,+;u) = \frac{Q_1(u+5)Q_2(u+2)Q_3(u+5)}{Q_1(u+3)Q_2(u+4)Q_3(u+3)}$$
$$sp(-,-,+,+;u) = \frac{Q_2(u+6)Q_4(u+3)}{Q_2(u+4)Q_4(u+5)}$$
$$sp(-,-,-,-;u) = \frac{Q_4(u+7)}{Q_4(u+5)}.$$

Here τ^Q is the operation $Q_a \mapsto Q_{a+1}$, that is

$$\tau^Q f(Q_1(u+x_1^1), Q_1(u+x_2^1), \ldots, Q_2(u+x_1^2), Q_2(u+x_2^2), \ldots)$$
$$= f(Q_2(u+x_1^1), Q_2(u+x_2^1), \ldots, Q_3(u+x_1^2), Q_3(u+x_2^2), \ldots) \tag{2.8}$$

for any function f. We shall use the functions $\mathcal{T}^a(u)$ for $a \in \mathbb{Z}$ and $u \in \mathbb{C}$ determined by the generating series

$$(1+z(\bar{1};u)X)\ldots(1+z(\overline{r-1};u)X)[-1+(1+z(r;u)X)(1-z(\bar{r};u)Xz(r;u)X)^{-1}$$
$$+(1+z(\bar{r};u)X)(1-z(r;u)Xz(\bar{r};u)X)^{-1}]$$
$$\times(1+z(r-1;u)X)\ldots(1+z(1;u)X) = \sum_{a=-\infty}^{\infty}\mathcal{T}^a(u+a-1)X^a \tag{2.9}$$

where X is a shift operator $X = \mathrm{e}^{2\partial_u}$. Namely, for $a < 0$ $\mathcal{T}^a(u) = 0$ and for $a \geqslant 0$,

$$\mathcal{T}^a(u) = \sum_{\substack{i_1<\cdots<i_k \\ j_1<\cdots<j_l \\ k+l+2n=a}} z(i_1; v_1)\ldots z(i_k; v_k) \times z(\bar{r}; v_{k+1}) z(r; v_{k+2})\ldots z(\bar{r}; v_{k+2n-1}) \times z(r; v_{k+2n}) z(\overline{j_l}; v_{k+2n+1})\ldots z(\overline{j_1}; v_a) \tag{2.10}$$

where $i_\alpha, j_\beta \in \{1, \ldots, r\}$, $k, l, n \in \mathbb{Z}_{\geqslant 0}$ and $v_\gamma = u + a - 2\gamma + 1$. We define the functions $T_1^{(a)}(u)$ for $1 \leqslant a \leqslant r$ that correspond to the dress parts of eigenvalue of transfer matrix in Kuniba and Suzuki (1995):

$$T_1^{(a)}(u) = \mathcal{T}^a(u) \qquad \text{for } 1 \leqslant a \leqslant r-2 \tag{2.11a}$$

$$T_1^{(r-1)}(u) = \sum_{(\xi_1,\ldots,\xi_r)\in \text{ Spin}^-} sp(\xi_1, \ldots, \xi_r; u) \tag{2.11b}$$

$$T_1^{(r)}(u) = \sum_{(\xi_1,\ldots,\xi_r)\in \text{ Spin}^+} sp(\xi_1, \ldots, \xi_r; u) \tag{2.11c}$$

where for $\epsilon = \pm$ we have put

$$\text{Spin}^\epsilon = \left\{(\xi_1, \ldots, \xi_r) : \xi_j = \pm, \prod_{j=1}^{r} \xi_j = \epsilon\right\}. \tag{2.12}$$

Theorem 2.1. (Kuniba and Suzuki 1995) For any integer a, $\mathcal{T}^a(u)$, $T_1^{(r-1)}(u)$ and $T_1^{(r)}(u)$ are free of poles under the condition that the BAE is valid.

Actually in Kuniba and Suzuki (1995) only the $a \leqslant r-2$ case was considered for $\mathcal{T}^a(u)$ but the proof therein is valid for any a. Except for this theorem, all the definitions and the statements in this paper make sense without assuming (2.1) and (2.2) as mentioned in the introduction.

Let us now explain the relations between $z(a; u)$ and $sp(\xi_1, \ldots, \xi_r; u)$. Define $i_1 < \cdots < i_k$, $I_1 < \cdots < I_{r-k}$ $(0 \leqslant k \leqslant r)$ and $j_1 < \cdots < j_l$, $J_1 < \cdots < J_{r-l}$ $(0 \leqslant l \leqslant r)$ using the two sequences $(\xi_1, \ldots, \xi_r)$ and $(\eta_1, \ldots, \eta_r) \in \{\pm\}^r$ as follows:

$$\begin{aligned} &\xi_{i_1} = \cdots = \xi_{i_k} = + \qquad \xi_{I_1} = \cdots = \xi_{I_{r-k}} = - \\ &\eta_{j_1} = \cdots = \eta_{j_l} = - \qquad \eta_{J_1} = \cdots = \eta_{J_{r-l}} = +. \end{aligned} \tag{2.13}$$

Using the relations (2.5), (2.6) and (2.7) and induction on r, we have

Proposition 2.2. For any $a \in \mathbb{Z}_{\geqslant 0}$,

$$\prod_{n=1}^{a} z(b_n; u + a + 1 - 2n) = sp(\xi_1, \ldots, \xi_r; u - r + a + 1) sp(\eta_1, \ldots, \eta_r; u + r - a - 1) \tag{2.14a}$$

$$b_n = \begin{cases} i_n & \text{for } 1 \leqslant n \leqslant k \\ r & \text{for } k < n \leqslant a - l \text{ and } n \equiv k \bmod 2 \\ \bar{r} & \text{for } k < n \leqslant a - l \text{ and } n \not\equiv k \bmod 2 \\ \overline{j_{a+1-n}} & \text{for } a - l < n \leqslant a \end{cases} \tag{2.14b}$$

if $k + l \leqslant a$ and $a \equiv l + k \bmod 2$. For any $a \in \mathbb{Z}_{\leqslant 2r-2}$,

$$\prod_{n=1}^{2r-a-2} z(b'_n; u + 2r - a - 1 - 2n) = sp(\xi_1, \ldots, \xi_r; u - r + a + 1)sp(\eta_1, \ldots, \eta_r; u + r - a - 1) \tag{2.15a}$$

$$b'_n = \begin{cases} J_n & \text{for } 1 \leqslant n \leqslant r - l \\ r & \text{for } r - l < n \leqslant r + k - a - 2 \text{ and } n - r + l \equiv 0 \bmod 2 \\ \bar{r} & \text{for } r - l < n \leqslant r + k - a - 2 \text{ and } n - r + l \equiv 1 \bmod 2 \\ \overline{I_{2r-a-1-n}} & \text{for } r + k - a - 2 < n \leqslant 2r - a - 2 \end{cases} \tag{2.15b}$$

if $k + l \geqslant a + 2$ and $a \equiv l + k \bmod 2$.

For $a \leqslant r - 2$, (2.14*a*) is equation (B.1) in Kuniba and Suzuki (1995). The following new functional relation is the D_r version of equation (2.14) in Kuniba *et al* (1995), which is derived by summing up the equations (2.14*a*) and (2.15*a*).

Proposition 2.3.

$$\mathcal{T}^a(u) + \mathcal{T}^{2r-a-2}(u) = T_1^{(r)}(u + r - a - 1)T_1^{(r-\delta_{r-a})}(u - r + a + 1) + T_1^{(r-1)}(u + r - a - 1)T_1^{(r-\delta_{r-a-1})}(u - r + a + 1) \tag{2.16}$$

where

$$\delta_i = \begin{cases} 0 & \text{if } i \in 2\mathbb{Z} \\ 1 & \text{if } i \in 2\mathbb{Z} + 1. \end{cases}$$

Note that (2.16) is invariant under the exchange $a \leftrightarrow 2r - 2 - a$; in particular

$$\mathcal{T}^{r-1}(u) = T_1^{(r)}(u)T_1^{(r-1)}(u). \tag{2.17}$$

3. Main results

Set

$$\begin{aligned} &x_j(u) = T_1^{(r-\delta_{j-1})}(u + 2j - 2) \qquad y_j(u) = T_1^{(r-\delta_j)}(u + 2j - 2) \\ &t_{ij}(u) = T_1^{(r+i-j-1)}(u + i + j - 2) \qquad a_{ij}(u) = x_i(u)y_j(u) - t_{ij}(u) \\ &b_{ij}(u) = y_i(u)x_j(u) - t_{ij}(u). \end{aligned} \tag{3.1}$$

By definition one has

$$\begin{aligned} &x_i(u + 2) = y_{i+1}(u) \qquad y_i(u + 2) = x_{i+1}(u) \qquad t_{ij}(u + 2) = t_{i+1\, j+1}(u) \\ &a_{ij}(u + 2) = b_{i+1\, j+1}(u) \qquad b_{ij}(u + 2) = a_{i+1\, j+1}(u). \end{aligned} \tag{3.2}$$

Now we introduce the $(m + 1) \times (m + 1)$ matrix $\mathcal{S}_{m+1}(u) = (\mathcal{S}_{ij})_{1\leqslant i,j\leqslant m+1}$ whose (i, j) elements are given by

$$\mathcal{S}_{ij} = \begin{cases} 0 & \text{for } i = j = 1 \\ T_1^{(r-\delta_j)}(u + 2j - 4) & \text{for } i = 1 \text{ and } 2 \leqslant j \leqslant m + 1 \\ -T_1^{(r-\delta_i)}(u + 2i - 4) & \text{for } 2 \leqslant i \leqslant m + 1 \text{ and } j = 1 \\ -\mathcal{T}^{r+i-j-1}(u + i + j - 4) & \text{for } 2 \leqslant i,\ j \leqslant m + 1. \end{cases} \tag{3.3}$$

Using the relation (2.11*a*) and (2.16), the matrix elements of $\mathcal{S}_{m+1}(u)$ can be rewritten as

$$\mathcal{S}_{ij} = \begin{cases} 0 & \text{for } i = j = 1 \\ x_{j-1} & \text{for } i = 1 \text{ and } 2 \leqslant j \leqslant m+1 \\ -x_{i-1} & \text{for } 2 \leqslant i \leqslant m+1 \text{ and } j = 1 \\ -x_{i-1}y_{i-1} & \text{for } i = j \text{ and } 2 \leqslant i \leqslant m+1 \\ -t_{i-1\,j-1} & \text{for } 2 \leqslant i < j \leqslant m+1 \\ t_{j-1\,i-1} - x_{j-1}y_{i-1} - x_{i-1}y_{j-1} & \text{for } 2 \leqslant j < i \leqslant m+1. \end{cases} \tag{3.4}$$

For $\mathcal{S}_{m+1}(u) = (\mathcal{S}_{ij})_{1\leqslant i,j\leqslant m+1}$, we introduce the following anti-symmetric matrices $\mathcal{C}_{m+1}(u)$ and $\mathcal{R}_{m+1}(u)$:

$$\mathcal{C}_{m+1}(u) := \mathcal{S}_{m+1}(u) \prod_{j=2}^{m+1} \mathbf{P}(1, j; -y_{j-1}(u))$$
$$= \begin{pmatrix} 0 & x_1 & x_2 & \ldots & x_m \\ -x_1 & 0 & a_{12} & \ldots & a_{1m} \\ -x_2 & -a_{12} & 0 & \ldots & a_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_m & -a_{1m} & -a_{2m} & \ldots & 0 \end{pmatrix} \tag{3.5a}$$

$$\mathcal{R}_{m+1}(u) := \left(\prod_{i=2}^{m+1} \mathbf{P}(i, 1; y_{i-1}(u))\right)\mathcal{S}_{m+1}(u)$$
$$= \begin{pmatrix} 0 & x_1 & x_2 & \ldots & x_m \\ -x_1 & 0 & b_{12} & \ldots & b_{1m} \\ -x_2 & -b_{12} & 0 & \ldots & b_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_m & -b_{1m} & -b_{2m} & \ldots & 0 \end{pmatrix}. \tag{3.5b}$$

Here

$$\mathbf{P}(i, j; c) = E + cE_{ij} \tag{3.6}$$

is the $(m+1) \times (m+1)$ matrix with E the identity and E_{ij} the matrix unit. The products of $\mathbf{P}$'s in the above are commutative. For any matrix $\mathcal{M}(u)$, we shall let

$$\mathcal{M}\begin{bmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_k \end{bmatrix}(u)$$

denote the minor matrix removing i_l's rows and j_l's columns from $\mathcal{M}(u)$. Our main results in this paper are given as follows.

Theorem 3.1. The following determinant and Pfaffian expressions solve the D_rT-system (1.1):

$$T_m^{(a)}(u) = \det_{1\leqslant i,j\leqslant m}[\mathcal{T}^{a+i-j}(u+i+j-m-1)] \qquad \text{for } a \in \{1, 2, \ldots, r-2\},\ m \in \mathbb{Z}_{\geqslant 0}$$
$$T_m^{(r)}(u) = \begin{cases} \mathrm{pf}\left[\mathcal{C}_{m+1}\begin{bmatrix} 1 \\ 1 \end{bmatrix}(u-m+1)\right] & \text{for } m \in 2\mathbb{Z}_{\geqslant 0} \\ \mathrm{pf}[\mathcal{C}_{m+1}(u-m+1)] & \text{for } m \in 2\mathbb{Z}_{\geqslant 0}+1 \end{cases} \tag{3.7a}$$

$$T_m^{(r-1)}(u) = \begin{cases} \mathrm{pf}\left[\mathcal{C}_{m+2}\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix}(u-m-1)\right] & \text{for } m \in 2\mathbb{Z}_{\geqslant 0} \\ \mathrm{pf}\left[\mathcal{C}_{m+2}\begin{bmatrix}2\\ 2\end{bmatrix}(u-m-1)\right] & \text{for } m \in 2\mathbb{Z}_{\geqslant 0}+1. \end{cases} \tag{3.7b}$$

4. Proof of theorem 3.1

At first, we present a number of lemmas that are necessary for the proof. The following Jacobi identity ($b \neq c$) plays an important role in this section:

$$\det \mathcal{M}\begin{bmatrix}b\\ b\end{bmatrix}\det \mathcal{M}\begin{bmatrix}c\\ c\end{bmatrix} - \det \mathcal{M}\begin{bmatrix}b\\ c\end{bmatrix}\det \mathcal{M}\begin{bmatrix}c\\ b\end{bmatrix} = \det \mathcal{M}\begin{bmatrix}b & c\\ b & c\end{bmatrix}\det \mathcal{M}. \tag{4.1}$$

Lemma 4.1. (Kuniba *et al* 1994.) For any $a, m \in \mathbb{Z}_{\geqslant 0}$ and $u \in \mathbb{C}$ put

$$\mathcal{T}_m^a(u) = \det_{1\leqslant i,j\leqslant m}[\mathcal{T}^{a+i-j}(u+i+j-m-1)]. \tag{4.2}$$

Then the following functional relation is valid:

$$\mathcal{T}_m^a(u-1)\mathcal{T}_m^a(u+1) = \mathcal{T}_{m+1}^a(u)\mathcal{T}_{m-1}^a(u) + \mathcal{T}_m^{a-1}(u)\mathcal{T}_m^{a+1}(u). \tag{4.3}$$

Proof. Apply (4.1) for $(b, c) = (1, m+1)$ to $\mathcal{M} = [\mathcal{T}^{a+i-j}(u+i+j-m-2)]_{1\leqslant i,j\leqslant m+1}$. □

Lemma 4.2. For (3.7*a*)–(3.7*c*) to satisfy (1.1*b*) it is enough to show

$$T_m^{(r-1)}(u)T_m^{(r)}(u) = \mathcal{T}_m^{r-1}(u). \tag{4.4}$$

Proof. From lemma 4.1 and (3.7*a*), we have $T_m^{(a)}(u) = \mathcal{T}_m^a(u)$ for $1 \leqslant a \leqslant r-2$. Then compare (1.1*b*) and (4.3) for $a = r-2$. □

By noting $\det[P(i, j; c)] = 1$, we have

Lemma 4.3.

$$\det[\mathcal{S}_{m+1}(u)] = \det[\mathcal{C}_{m+1}(u)] = \det[\mathcal{R}_{m+1}(u)]. \tag{4.5}$$

We shall further need

Lemma 4.4. For $m \in \mathbb{Z}_{\geqslant 0}$, $T_m^{(r-1)}(u)$ (3.7*c*) and $T_m^{(r)}(u)$ (3.7*b*) satisfy the following relations:

$$T_m^{(r-1)}(u+1)T_{m-1}^{(r)}(u) = \begin{cases} \det\left[\mathcal{S}_{m+1}\begin{bmatrix}m+1\\ 1\end{bmatrix}(u-m+2)\right] & \text{for } m \in 2\mathbb{Z}_{\geqslant 0} \\ \det\left[\mathcal{S}_{m+2}\begin{bmatrix}2 & m+2\\ 1 & 2\end{bmatrix}(u-m)\right] & \text{for } m \in 2\mathbb{Z}_{\geqslant 0}+1 \end{cases} \tag{4.6a}$$

$$T_m^{(r-1)}(u)T_m^{(r)}(u) = (-1)^m \det\left[\mathcal{S}_{m+1}\begin{bmatrix}1\\ 1\end{bmatrix}(u-m+1)\right] \tag{4.6b}$$

$$T_m^{(r-1)}(u+1)T_{m+1}^{(r)}(u) = (-1)^{m+1} \det\left[\mathcal{S}_{m+2}\begin{bmatrix}1\\ 2\end{bmatrix}(u-m)\right] \tag{4.6c}$$

$$T_{m-1}^{(r-1)}(u)T_m^{(r-1)}(u+1) = (-1)^m \det\left[\mathcal{S}_{m+2}\begin{bmatrix}1 & 2\\ 2 & m+2\end{bmatrix}(u-m)\right] \tag{4.6d}$$

$$T_{m-1}^{(r)}(u-1)T_m^{(r)}(u) = (-1)^m \det\left[\mathcal{S}_{m+1}\begin{bmatrix}1\\ m+1\end{bmatrix}(u-m+1)\right] \tag{4.6e}$$

$$T_m^{(r-1)}(u)T_{m-1}^{(r)}(u+1) = (-1)^m \det\left[\mathcal{S}_{m+2}\begin{bmatrix}1 & 2\\ 2 & 3\end{bmatrix}(u-m-1)\right]. \tag{4.6f}$$

Proof. All the relations in lemma 4.4 reduce to the Jacobi identity. First we prove (4.6*a*) for $m \in 2\mathbb{Z}_{\geqslant 0}$. Let $\mathcal{M} = \mathcal{R}_{m+1}(u-m+2)$ and, noting the relation (4.5), we have

$$\begin{aligned}
&\det\mathcal{M} = \det[\mathcal{R}_{m+1}(u-m+2)] = 0 \qquad \det\mathcal{M}\begin{bmatrix}1\\1\end{bmatrix} = (T_m^{(r-1)}(u+1))^2\\
&\det\mathcal{M}\begin{bmatrix}m+1\\m+1\end{bmatrix} = (T_{m-1}^{(r)}(u))^2\\
&\det\mathcal{M}\begin{bmatrix}1\\m+1\end{bmatrix} = \det\mathcal{M}\begin{bmatrix}m+1\\1\end{bmatrix} = \det\left[\mathcal{S}_{m+1}\begin{bmatrix}m+1\\1\end{bmatrix}(u-m+2)\right].
\end{aligned} \tag{4.7}$$

The first identity follows from the fact that the determinant of antisymmetric matrix of odd size should vanish. The others follow from (3.7*b*), (3.7*c*) and (3.2). Substituting these identities into (4.1) for $(b, c) = (1, m+1)$, we have

$$(T_m^{(r-1)}(u+1)T_{m-1}^{(r)}(u))^2 = \left(\det\left[\mathcal{S}_{m+1}\begin{bmatrix}m+1\\1\end{bmatrix}(u-m+2)\right]\right)^2. \tag{4.8}$$

Taking the square root of (4.8), we have (4.6*a*) for $m \in 2\mathbb{Z}_{\geqslant 0}$. The relative sign can be determined so that the equation is valid for $m = 0$ and 2 or more rigorously, by comparing the sign of the coefficient of $x_1(u-m+2)\ldots\cdot x_m(u-m+2)\cdot y_1(u-m+2)\ldots y_{m-1}(u-m+2)$ on both sides. The other identities can be proved by a similar method. Here we list $\mathcal{M}$ and (b, c) to be used in (4.1) and some other relations particularly needed. Equations (3.2) and (4.5) should also be used.

(4.6*a*) for $m \in 2\mathbb{Z}_{\geqslant 0}+1$:

$$\mathcal{M} = \mathcal{R}_{m+2}\begin{bmatrix}2\\2\end{bmatrix}(u-m) \qquad \text{with } (b, c) = (1, m+1).$$

(4.6*b*) for $m \in 2\mathbb{Z}_{\geqslant 0}+1$:

$$\mathcal{M} = \begin{pmatrix} 0 & -1 & y_1 & y_2 & \ldots & y_m\\ 1 & 0 & x_1 & x_2 & \ldots & x_m\\ -y_1 & -x_1 & 0 & a_{12} & \ldots & a_{1m}\\ -y_2 & -x_2 & -a_{12} & 0 & \ldots & a_{2m}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ -y_m & -x_m & -a_{1m} & -a_{2m} & \ldots & 0\end{pmatrix}(u-m+1) \tag{4.9}$$

with $(b, c) = (1, 2)$.

(4.6*c*):

$$\mathcal{M} = \mathcal{C}_{m+2}(u-m) \qquad \text{with } (b, c) = (1, 2).$$

(4.6*d*):

$$\mathcal{M} = \mathcal{C}_{m+2}\begin{bmatrix}2\\2\end{bmatrix}(u-m) \qquad \text{with } (b, c) = (1, m+1).$$

(4.6e):

$$\mathcal{M} = \mathcal{C}_{m+1}(u-m+1) \qquad \text{with } (b,c) = (1, m+1).$$

(4.6b) for $m \in 2\mathbb{Z}_{\geqslant 0}$:

$$\mathcal{M} = \mathcal{S}_{m+1}(u-m+1) \qquad \text{with } (b,c) = (1, m+1)$$

and the relations (3.7b), (4.6e) and (4.6a).

(4.6f):

$$\mathcal{M} = \mathcal{C}_{m+2}\begin{bmatrix}2\\2\end{bmatrix}(u-m-1) \qquad \text{with } (b,c) = (1, 2).$$

□

We have presented similar relations for (3.7c) and (3.7b) in the appendix.

Proof of theorem 3.1. Equation (1.1a) follows from lemma 4.1 and (1.1b) from lemma 4.2 and (4.6b). Equation (1.1c) for $a = r$ is derived as follows. Let

$$\mathcal{M} = \mathcal{S}_{m+2}\begin{bmatrix}1\\2\end{bmatrix}(u-m)$$

then from (4.6c), (4.6b), (4.6d) and (3.7a), we have

$$\begin{aligned}
\det\mathcal{M} &= (-1)^{m+1}T_m^{(r-1)}(u+1)T_{m+1}^{(r)}(u)\\
\det\mathcal{M}\begin{bmatrix}1\\1\end{bmatrix} &= (-1)^{m}T_m^{(r-1)}(u+1)T_{m}^{(r)}(u+1)\\
\det\mathcal{M}\begin{bmatrix}m+1\\m+1\end{bmatrix} &= (-1)^{m}T_{m-1}^{(r-1)}(u)T_{m}^{(r)}(u-1)\\
\det\mathcal{M}\begin{bmatrix}1\\m+1\end{bmatrix} &= (-1)^{m}T_m^{(r-1)}(u+1)T_{m-1}^{(r-1)}(u)\\
\det\mathcal{M}\begin{bmatrix}m+1\\1\end{bmatrix} &= (-1)^{m}T_m^{(r-2)}(u)\\
\det\mathcal{M}\begin{bmatrix}1 & m+1\\1 & m+1\end{bmatrix} &= (-1)^{m-1}T_{m-1}^{(r-1)}(u)T_{m-1}^{(r)}(u)
\end{aligned} \tag{4.10}$$

Applying (4.1) for $b = 1$ and $c = m+1$ to (4.10), we get (1.1c) for $a = r$. Equation (1.1c) for $a = r-1$ is derived quite similarly. Let

$$\mathcal{M} = \mathcal{S}_{m+3}\begin{bmatrix}1 & 2\\2 & 3\end{bmatrix}(u-m-2)$$

then from (4.6b), (4.6f), (3.7a) and (4.6e), we have

$$\begin{aligned}
\det\mathcal{M} &= (-1)^{m+1}T_{m+1}^{(r-1)}(u)T_{m}^{(r)}(u+1)\\
\det\mathcal{M}\begin{bmatrix}1\\1\end{bmatrix} &= (-1)^{m}T_m^{(r-1)}(u+1)T_{m}^{(r)}(u+1)\\
\det\mathcal{M}\begin{bmatrix}m+1\\m+1\end{bmatrix} &= (-1)^{m}T_{m}^{(r-1)}(u-1)T_{m-1}^{(r)}(u)\\
\det\mathcal{M}\begin{bmatrix}1\\m+1\end{bmatrix} &= (-1)^{m}T_{m-1}^{(r)}(u)T_{m}^{(r)}(u+1)
\end{aligned} \tag{4.11}$$

$$\det \mathcal{M}\begin{bmatrix} m+1 \\ 1 \end{bmatrix} = (-1)^m T_m^{(r-2)}(u)$$

$$\det \mathcal{M}\begin{bmatrix} 1 & m+1 \\ 1 & m+1 \end{bmatrix} = (-1)^{m-1} T_{m-1}^{(r-1)}(u) T_{m-1}^{(r)}(u).$$

Applying (4.1) for $b = 1$ and $c = m + 1$ to (4.11), we have (1.1c) for $a = r - 1$. □

Remark. Reflecting the Dynkin diagram symmetry of D_r, similar relations to lemma 4.4, theorem 3.1 and those in the appendix can be obtained by exchanging $T_1^{(r-1)}(u)$ and $T_1^{(r)}(u)$.

5. Discussion

In this paper, we have given a new representation of the solution to the D_r T-system (1.1). The key is the introduction of the auxiliary dress function $\mathcal{T}^a$ (2.9) and the new functional relation (2.16). These are motivated from the analytic Bethe ansatz and lead to a different expression of the solution from the earlier one (Kuniba *et al* 1996).

A similar analysis has been performed in Kuniba *et al* (1995) for the B_r case. There, a more general class of transfer matrix spectra has been represented not only by determinants but also as summations over certain tableaux. These are B_r Yangian analogues of the semi-standard Young tableaux for $sl(r+1)$. There remains a problem to extend such an analysis to the D_r case. So far we have only found a conjecture on the tableau sum representations of $T_m^{(r)}(u)$ and $T_m^{(r-1)}(u)$, as stated below.

Consider an injection $\iota : \mathrm{Spin}^\epsilon \to J^r$, sending $(\zeta_1, \ldots, \zeta_r)$ to $(i_1, \ldots, i_k, \overline{j_{r-k}}, \ldots, \overline{j_1})$ such that $\zeta_{i_1} = \cdots = \zeta_{i_k} = +$, $\zeta_{j_1} = \cdots = \zeta_{j_{r-k}} = -$, $1 \leqslant i_1 < \cdots < i_k \leqslant r$ and $1 \leqslant j_1 < \cdots < j_{r-k} \leqslant r$. We shall write the components as $\iota(\zeta) = (\iota(\zeta)_1, \ldots, \iota(\zeta)_r)$. For $\epsilon = \pm$ and $m \in \mathbb{Z}_{\geqslant 1}$ put

$$\mathrm{Spin}_m^\epsilon = \{(\zeta^{(1)}, \ldots, \zeta^{(m)}) \in (\mathrm{Spin}^\epsilon)^m : \iota(\zeta^{(i)})_a \preceq \iota(\zeta^{(i+1)})_a \quad \text{for } 1 \leqslant i \leqslant m-1,\ 1 \leqslant a \leqslant r\}. \tag{5.1}$$

This is well defined because the situations $(\iota(\zeta^{(i)})_a, \iota(\zeta^{(i+1)})_a) = (r, \bar{r})$ and $(\bar{r}, r)$ never happen due to the parity constraint in (2.12). In particular $\mathrm{Spin}_1^\epsilon = \mathrm{Spin}^\epsilon$. Now our conjecture reads

$$T_m^{(r+(\epsilon-1)/2)}(u) = \sum_{(\zeta^{(1)},\ldots,\zeta^{(m)}) \in\ \mathrm{Spin}_m^\epsilon} \prod_{i=1}^m sp(\zeta^{(i)}; u - m + 2i - 1). \tag{5.2}$$

We have verified this for $4 \leqslant r \leqslant 6$, $1 \leqslant m \leqslant 2$.

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Appendix. Other relations

The following relations are valid.

$$T_m^{(r-1)}(u+1) T_m^{(r)}(u-1) = \begin{cases} \det\left[\mathcal{C}_{m+2}\begin{bmatrix} 1 & 2 \\ 1 & m+2 \end{bmatrix}(u-m)\right] & \text{for } m \in 2\mathbb{Z}_{\geqslant 0} \\ \det\left[\mathcal{S}_{m+2}\begin{bmatrix} 2 \\ m+2 \end{bmatrix}(u-m)\right] & \text{for } m \in 2\mathbb{Z}_{\geqslant 0}+1 \end{cases} \tag{A.1}$$

$$T^{(r-1)}_{m-1}(u)T^{(r)}_{m}(u+1)=\begin{cases}\det\left[\mathcal{S}_{m+2}\begin{bmatrix}2 & m+2\\ 1 & 2\end{bmatrix}(u-m)\right] & \text{for } m\in 2\mathbb{Z}_{\geqslant 0}\\ \det\left[\mathcal{S}_{m+1}\begin{bmatrix}m+1\\ 1\end{bmatrix}(u-m+2)\right] & \text{for } m\in 2\mathbb{Z}_{\geqslant 0}+1\end{cases} \tag{A.2}$$

$$T^{(r-\delta_m)}_{m+1}(u)T^{(r-\delta_m)}_{m}(u+1)=\det\left[\mathcal{S}_{m+2}\begin{bmatrix}2\\ 1\end{bmatrix}(u-m)\right] \tag{A.3}$$

$$T^{(r-\delta_{m-1})}_{m+1}(u)T^{(r-\delta_{m-1})}_{m}(u+1)=\det\left[\mathcal{S}_{m+3}\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}(u-m-2)\right] \tag{A.4}$$

$$T^{(r-1)}_{m}(u-1)T^{(r)}_{m}(u+1)=\begin{cases}\det\left[\mathcal{C}_{m+3}\begin{bmatrix}1 & 2 & m+3\\ 1 & 2 & 3\end{bmatrix}(u-m-2)\right]\\ \qquad\text{for } m\in 2\mathbb{Z}_{\geqslant 0}\\ \det\left[\mathcal{S}_{m+3}\begin{bmatrix}2 & m+3\\ 2 & 3\end{bmatrix}(u-m-2)\right]\\ \qquad\text{for } m\in 2\mathbb{Z}_{\geqslant 0}+1\end{cases} \tag{A.5}$$

$$T^{(r-1)}_{m+1}(u)T^{(r)}_{m-1}(u)=\begin{cases}\det\left[\mathcal{S}_{m+3}\begin{bmatrix}2 & m+3\\ 2 & 3\end{bmatrix}(u-m-2)\right] & \text{for } m\in 2\mathbb{Z}_{\geqslant 0}\\ \det\left[\mathcal{C}_{m+3}\begin{bmatrix}1 & 2 & m+3\\ 1 & 2 & 3\end{bmatrix}(u-m-2)\right] & \text{for } m\in 2\mathbb{Z}_{\geqslant 0}+1\end{cases} \tag{A.6}$$

$$T^{(r-1)}_{m-1}(u)T^{(r)}_{m+1}(u)=\begin{cases}\det\left[\mathcal{S}_{m+2}\begin{bmatrix}m+2\\ 2\end{bmatrix}(u-m)\right] & \text{for } m\in 2\mathbb{Z}_{\geqslant 0}\\ \det\left[\mathcal{C}_{m+2}\begin{bmatrix}1 & m+2\\ 1 & 2\end{bmatrix}(u-m)\right] & \text{for } m\in 2\mathbb{Z}_{\geqslant 0}+1.\end{cases} \tag{A.7}$$

Proof. The proof is performed in the same way as lemma 4.4. Here we list $\mathcal{M}$ and (b, c) to be used in (4.1) and some other relations particularly needed. Equations (4.5) and (3.2) should also be used when necessary.

(A.1) for $m \in 2\mathbb{Z}_{\geqslant 0}$:

$$\mathcal{M}=\mathcal{C}_{m+2}\begin{bmatrix}1\\ 1\end{bmatrix}(u-m) \qquad \text{with } (b,c)=(1,m+1)$$

for $m \in 2\mathbb{Z}_{\geqslant 0}+1$:

$$\mathcal{M}=\mathcal{S}_{m+2}(u-m) \qquad \text{with } (b,c)=(2,m+2).$$

(A.2) for $m \in 2\mathbb{Z}_{\geqslant 0}$:

$$\mathcal{M}=\mathcal{R}_{m+2}\begin{bmatrix}2\\ 2\end{bmatrix}(u-m) \qquad \text{with } (b,c)=(1,m+1)$$

for $m \in 2\mathbb{Z}_{\geqslant 0}+1$:

$$\mathcal{M}=\mathcal{R}_{m+1}(u-m+2) \qquad \text{with } (b,c)=(1,m+1).$$

(A.3) :

$$\mathcal{M}=\mathcal{S}_{m+2}(u-m) \qquad \text{with } (b,c)=(1,2)$$

and the relations (4.6*b*) and (4.6*c*).

(A.4) :

$$\mathcal{M} = \mathcal{S}_{m+3}\begin{bmatrix}2\\2\end{bmatrix}(u-m-2) \qquad \text{with } (b,c)=(1,2)$$

and the relations (4.6b) and (4.6f).

(A.5) for $m \in 2\mathbb{Z}_{\geqslant 0}$:

$$\mathcal{M} = \mathcal{C}_{m+3}\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}(u-m-2) \qquad \text{with } (b,c)=(1,m+1)$$

for $m \in 2\mathbb{Z}_{\geqslant 0}+1$:

$$\mathcal{M} = \mathcal{C}_{m+3}\begin{bmatrix}2\\2\end{bmatrix}(u-m-2)\text{with } (b,c)=(2,m+2).$$

(A.6) for $m \in 2\mathbb{Z}_{\geqslant 0}$:

$$\mathcal{M} = \mathcal{C}_{m+3}\begin{bmatrix}2\\2\end{bmatrix}(u-m-2) \qquad \text{with } (b,c)=(2,m+2)$$

for $m \in 2\mathbb{Z}_{\geqslant 0}+1$:

$$\mathcal{M} = \mathcal{C}_{m+3}\begin{bmatrix}1 & 2\\1 & 2\end{bmatrix}(u-m-2) \qquad \text{with } (b,c)=(1,m+1).$$

(A.7) for $m \in 2\mathbb{Z}_{\geqslant 0}$:

$$\mathcal{M} = \mathcal{S}_{m+2}(u-m) \qquad \text{with } (b,c)=(2,m+2)$$

for $m \in 2\mathbb{Z}_{\geqslant 0}+1$:

$$\mathcal{M} = \mathcal{C}_{m+2}\begin{bmatrix}1\\1\end{bmatrix}(u-m) \qquad \text{with } (b,c)=(1,m+1).$$

□

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